— Exercises —

1. **Local vs. global diffeomorphism I.** For each of the following f: show that f is a local diffeomorphism, i.e., $\forall p$ in the domain, $\exists B$ a neighborhood of p such that $f_{|B}$ is a differentiable bijection onto its image, whose inverse is also differentiable. Discuss the injectivity and the surjectivity of f.

   (a)
$$\begin{aligned} f : \mathbb{R}^2 - \{(0,0)\} &\to \mathbb{R}^2 - \{(0,0)\} \\ (x,y) &\mapsto (x^2 - y^2, 2xy). \end{aligned}$$

   (b)
$$\begin{aligned} f : \quad \mathbb{R}^3 &\to \mathbb{R}^3 \\ (x,y,z) &\mapsto (e^{2y} + e^{2z}, e^{2x} - e^{2z}, x - y). \end{aligned}$$

2. **Application of the inverse function theorem.** Let U be an open set in $\mathbb{R}^n$ and f a $\mathcal{C}^1$ map from U into $\mathbb{R}^n$. We assume that, for every c in U, $df(c)$ is inversible. Show that f is an open map (that is $f(O)$ is open for every open set contained in U).

— Problem —

3. **Local vs. global diffeomorphism II.** Let $a, b \in \mathbb{R}$ and
$$\begin{aligned} f : \quad \mathbb{R}^2 &\to \mathbb{R}^2 \\ (x,y) &\mapsto (x + a\sin y, y + b\sin x). \end{aligned}$$

   (a) Compute the Jacobian matrix of f at every point.

   (b) What is the condition on a and b for the assumption of the Inverse Function Theorem to hold at every (x, y)? In what follows, we assume that this condition holds.

   (c) Show that $\forall t, t' \in \mathbb{R}, |\sin t - \sin t'| \le |t - t'|$. Deduce that f is injective.

   (d) Show that f is actually a bijection by showing that its image is open and closed.